

Module-1

// -> Degree Naming

1st Linear

2nd Quadratic

3rd Cubic

4th Bi-Quadratic

5th Quintic

6th Hexic

7th Heptic

8th Octic

9th Nonic

10th Decic

//→ An expressⁿ in form of $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where $a_i \in \mathbb{R}$ or constants & (n) is a +ve integer & provided $a_n \neq 0$ is called a polynomial of degree (n) in 1 variable (x) .

//→ Transcendental eqns:-

[If an eqnⁿ includes logarithmic, trigonometric, exponential or inv. trigonometric functⁿ, then its called a transcendental eqnⁿ. e.g.

$$\cos x + xe^x = 0$$

//→ Numerical approx. method / Iterative method:-

These methods are based ~~on~~ on the idea of successive approximatⁿ, i.e. starting with 1 or 2 initial approximatⁿ to the exact root, certain mathematical conditⁿ or formula is repeatedly applied under similar mathematical conditⁿ to generate a sequence of nos. which are successive approximatⁿs to the solutⁿs of the problem which is in the limit converges to the root.

//→ Intermediate value th^m:-

[If $f(x)$ is a continuous functⁿ in an interval (a, b) & $f(a) \cdot f(b)$ is -ve i.e. $f(a) \cdot f(b) < 0$ i.e. $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ has atleast 1 real root or an odd no. of roots lying in the interval (a, b) .

Q: $f(x) = x^3 - 5x + 1 = 0$. Find out the interval in which the root of the given equn. lies.

Ans- $f(0) = 1$
 $f(1) = -3 \Rightarrow$ Root lies in $(0, 1)$

Between 0 & 1, 0 is the nearer root.

$f(0.1) = 0.501 > 0$

$f(0.2) = 0.008 > 0$

$f(0.3) = -0.473 < 0$

So, root lies in $(0.2, 0.3)$

Q: $f(x) = 8x^3 - 12x^2 + 6x - 1 = 0$

Ans- $f(0) = -1 < 0$
 $f(1) = 1 > 0 \Rightarrow$ Root lies in $(0, 1)$

Q: $f(x) = \cos x - xe^{2x} = 0$

Ans- $f(0) = 1 - 0 = 1 > 0$

$f(1) = -2.1779 < 0$

Root lies between $(0, 1)$

$f(0.5) = 0.053 > 0$

$f(0.6) = -0.267 < 0$

Q: $f(x) = \sin x - 10x + 10 = 0$. Find the interval in which the roots lie.

Ans- $f(0) = 10 > 0$

$f(1) = 0.84 > 0$

$f(2) = -9.09 < 0$

Range lies in $(1, 2)$

$f(1.09) = -0.013 < 0$

$f(1.08) = 0.082 > 0$

∴ the roots lie in $(1.08, 1.09)$

// -> Numerical approx. method to solve non-linear eqns. -

2p (i) Bisection method

2p (ii) Regula falsi method (False Positⁿ Method)

2p (iii) Secant method

2p (iv) Newton Raphson method

2p (v) Fixed pt. iteratⁿ method

// -> Bisection Method -> [This method is based on the repeated applicatⁿ of intermediate value th^m. If we know that the root of $f(x)=0$ lies in the interval $I_0=(a, b)$, we bisect I_0 at the mid-pt. & denote the interval (a, m_0) if $f(a) \cdot f(m_0) < 0$ or (m_0, b) if $f(m_0) \cdot f(b) < 0$ & denoted as I_1 .

Again we bisect the interval I_1 & get another sub-interval I_2 at whose end pt. $f(x)$ takes the values of opp. signs & therefore contains the root.

Continuing this procedure several times, we obtain a sequence of nested set of subintervals. $I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$ such that each subinterval contains the root.

[After repeating the bisection process (n) times, we either find the root or find the interval (I_n) of length $\frac{b-a}{2^n}$.

[This method is simple but slowly convergent to the root. This method is linearly convergent.]

Q: Find an approx. to $\sqrt{3}$ correct upto 2 decimal places using bisection method.

Ans- So, eqn will be $\Rightarrow x^2 - 3 = 0 = f(x)$

Now, $f(0) = -3 < 0$

$$f(1) = -2 < 0$$

$$f(2) = 1 > 0$$

So, root lies in $(1, 2)$

Now,

$$f(1.5) = -0.75 < 0$$

$$f(1.6) = -0.44 < 0$$

$$f(1.7) = -0.11 < 0$$

$$f(1.8) = 0.24 > 0$$

So, root lies in $(1.7, 1.8)$

Here, let's consider the initial approx.

i.e. $x_0 = 1.7$ $x_1 = 1.8$
(-ve) (+ve)

① 1st:

$$x_2 = \frac{1.7 + 1.8}{2} = 1.75$$

$$\therefore f(1.75) = 0.0625 > 0$$

$$\therefore (x_0, x_2)$$

- +

② 2nd:

$$x_3 = \frac{1.7 + 1.75}{2} = 1.725$$

$$\therefore f(1.725) = -0.0243 < 0$$

$$\therefore (x_3, x_2)$$

- +

③ 3rd:

$$x_4 = \frac{1.725 + 1.75}{2} = 1.7375$$

$$\therefore f(1.7375) = 0.0189 > 0$$

$$\therefore (x_3, x_4)$$

- +

④ 4th:

$$x_5 = \frac{1.725 + 1.7375}{2} = 1.73125$$

$$\therefore f(1.73125) = -0.00277 < 0$$

So, approx. value will be 1.73

Q: Find the approx. ~~root~~ root of $f(x)$
 $= x - e^{-x} = 0$, performing 5 steps.

Ans. $f(0) = -1 < 0$

$$f(4) = 0.632 > 0$$

$$f(2) = 1.865 > 0$$

So, the root lies in $(0, 4)$

$$f(0.5) = -0.11 < 0 \quad] \quad x_0$$

$$f(0.6) = 0.05 > 0 \quad] \quad x_1$$

So, the root lies in $(0.5, 0.6)$

⊙ 1st:

$$x_2 = \frac{0.5 + 0.6}{2} = 0.55$$

$$\therefore f(0.55) = -0.027 < 0$$

$$\therefore (x_2, x_1)$$

- +

⊙ 2nd:

$$x_3 = \frac{0.55 + 0.6}{2} = 0.575$$

$$\therefore f(0.575) = 0.012 > 0$$

$$\therefore (x_2, x_3)$$

- +

⊙ 3rd:

$$x_4 = \frac{0.55 + 0.575}{2} = 0.5625$$

$$\therefore f(0.5625) = -0.007 < 0$$

$$\therefore (x_4, x_3)$$

- +

Q) 4th:

$$x_5 = \frac{0.5625 + 0.575}{2} = 0.56875$$

$$\therefore f(0.56875) = 0.00251 > 0$$

$$\therefore (x_5, x_4)$$

- +

Q) 5th:

$$x_6 = \frac{0.56875 + 0.563}{2} = 0.5658$$

$$\therefore f(0.5658) = -0.0021 < 0$$

Q) Find the approx. value of $\sqrt{2}$ upto 2 decimal places.

Ans: $f(x) = x^2 - 2 = 0$

$$\therefore f(0) = -2 < 0$$

$$f(1) = -1 < 0$$

$$f(2) = 2 > 0$$

$$f(-1) = -1 < 0$$

$\therefore$ the root lies in $(1, 2)$

Again, $f(1.4) = -0.04 < 0$

$$f(1.5) = 0.25 > 0$$

$\therefore$ the root lies in $(1.4, 1.5)$

Q) 1st:

$$x_2 = \frac{1.4 + 1.5}{2} = \frac{2.9}{2} = 1.45$$

$$\therefore f(1.45) = 0.1025 > 0$$

$$\therefore (x_0, x_2)$$

- +

Q) 2nd:

$$x_3 = \frac{1.4 + 1.45}{2} = \frac{2.85}{2} = 1.425$$

$$\therefore f(1.425) = 0.030625 > 0$$

$$\therefore (x_0, x_3)$$

$\begin{array}{c} - \quad + \end{array}$

② 3rd:

$$x_4 = \frac{1.4 + 1.425}{2} = 1.4125$$

$$\therefore f(1.4125) = -0.0048 < 0$$

$$\therefore (x_4, x_3)$$

So, the approx value is 1.41

11-→ Regula Falsi Method:

As in bisection method, in this method also, we need 2 initial approximates (a, b) , such that $f(a) \cdot f(b) < 0$. Use the following formula to find approximates:

$$x_1 = a - \left[f(a) \left\{ \frac{b - a}{f(b) - f(a)} \right\} \right]$$

Check for the sign of $f(x_1)$. Then according to $f(a) \cdot f(x_1) < 0$ or $f(x_1) \cdot f(b) < 0$, choose the 2nd interval.

To find the 2nd approx, use:

$$x_2 = a - \left[f(a) \left\{ \frac{x_1 - a}{f(x_1) - f(a)} \right\} \right]$$

[if the root lie in (a, x_1)]

- OR -

$$x_2 = b - \left[f(b) \left\{ \frac{b - x_1}{f(b) - f(x_1)} \right\} \right]$$

[if the root lie in (x_1, b)]

N.B: (i) If the 1st approx. lies to the right of the actual root, then all the successive approximations also lie to the right of it. In this case, the iterations are given by: (Right)

$$x_{k+1} = a - \left[f(a) \left\{ \frac{x_k - a}{f(x_k) - f(a)} \right\} \right]$$

$$k \in (1, 2, \dots)$$

(ii) If the 1st approx. lies to the left of the actual root, then the successive approx. also lie on the same side of it. So, the iterations are given by: (Left)

$$x_{k+1} = b - \left[f(b) \left\{ \frac{b - x_k}{f(b) - f(x_k)} \right\} \right]$$

$$k \in (1, 2, \dots)$$

Q: Find the root approx. of $f(x) = x^2 - 3 = 0$ by Regula Falsi Method.

$$\text{Ans: } \left. \begin{array}{l} f(1) = -2 < 0 \\ f(2) = 1 > 0 \end{array} \right\} \begin{array}{l} a \\ b \end{array}$$

$\therefore$ Root lies in $(1, 2) \therefore a = 1$ & $b = 2$

(i) 1st approx:-

$$x_1 = a - \left[f(a) \left\{ \frac{b - a}{f(b) - f(a)} \right\} \right]$$

$$= 1.6666667$$

$$\therefore f(1.6666667) = -0.222222 < 0$$

$\therefore$ Root lies in (x_1, b)

(ii) 2nd approx:-

$$x_2 = b - \left[f(b) \left\{ \frac{b - x_1}{f(b) - f(x_1)} \right\} \right] = 1.727272$$

$$\therefore f(x_2) = -0.01653 < 0$$

③ 3rd approx:-

$$x_3 = b - \left[f(b) \left\{ \frac{b - x_2}{f(b) - f(x_2)} \right\} \right]$$

$$= 1.731706$$

$$\therefore f(x_3) = -0.0011943 < 0$$

④ 4th approx:-

$$x_4 = b - \left[f(b) \left\{ \frac{b - x_3}{f(b) - f(x_3)} \right\} \right]$$

$$= 2 - \left[1 \left\{ \frac{2 - 1.731706}{1 + 0.0011943} \right\} \right]$$

$$= 1.7320260413$$

//→ Secant Method:-

This method is a modification of regula falsi method.

N.B: The rate of convergence of regula falsi method is also 1 i.e. it's also linearly convergent method.

The formula for finding out the root of the eqn. $f(x) = 0$ is given by:

$$x_{k+1} = x_k - \left[f(x_k) \left\{ \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right\} \right]$$

Let us consider the 1st 2 initial approx. as x_0 & x_1 . Then the 1st approx is given by:

$$x_2 = x_1 - \left[f(x_1) \left\{ \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right\} \right]$$

of the 2nd approx. is given by:

$$x_3 = x_2 - \left[f(x_2) \left\{ \frac{x_2 - x_1}{f(x_2) - f(x_1)} \right\} \right]$$

and so on...

The rate of convergence of secant method is $1.618 = \frac{1+\sqrt{5}}{2}$

This method is faster than bisection & regula falsi method.

Q: Find the root approx. of $f(x) = x^3 - 5x + 1 = 0$ using secant method upto 3 decimal places.

$$\begin{aligned} \text{Ans- } x_0 &= 0.2 & \Rightarrow f(x_0) &= 0.008 \\ x_1 &= 0.3 & \Rightarrow f(x_1) &= -0.473 \end{aligned}$$

Ⓐ 1st approx:

$$x_2 = x_1 - \left[f(x_1) \left\{ \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right\} \right]$$

$$= 0.3 - \left[(-0.473) \left\{ \frac{0.3 - 0.2}{-0.473 - 0.008} \right\} \right]$$

$$= 0.204663$$

$$\therefore f(x_2) = -0.000113$$

Ⓐ 2nd approx:

$$x_3 = x_2 - \left[f(x_2) \left\{ \frac{x_2 - x_1}{f(x_2) - f(x_1)} \right\} \right]$$

$$= 0.204663 - \left[(-0.000113) \right]$$

$$\left\{ \frac{0.204663 - 0.3}{-0.000113 + 0.473} \right\}$$

$$= 0.2016395$$

$$f(x_2) = 0.00000085718$$

Q: Find the root approx. of the eqn. of $f(x) = \cos x - x e^x = 0$ correct upto 3 decimal places using secant method.

$$\text{Ans} - x_0 = 0.5 \Rightarrow f(x_0) = 0.053221$$

$$x_1 = 0.6 \Rightarrow f(x_1) = -0.268$$

① 1st approx

$$x_2 = x_1 - \left[f(x_1) \left\{ \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right\} \right]$$

$$= 0.6 - \left[(-0.268) \left\{ \frac{0.6 - 0.5}{-0.268 - 0.053221} \right\} \right]$$

$$= 0.516568$$

$$\therefore f(x_2) = \cancel{0.000000118} 0.003614$$

② 2nd approx

$$x_3 = x_2 - \left[f(x_2) \left\{ \frac{x_2 - x_1}{f(x_2) - f(x_1)} \right\} \right]$$

$$= 0.516568 - \left[0.003614 \left\{ \frac{0.516568 - 0.6}{0.003614 + 0.268} \right\} \right]$$

$$= 0.517559$$

$$\therefore f(x_3) = 0.0006033$$

③ 3rd approx

$$x_4 = x_3 - \left[f(x_3) \left\{ \frac{x_3 - x_2}{f(x_3) - f(x_2)} \right\} \right]$$

$$= 0.517559 - \left[\begin{array}{l} 0.0006033 \\ \left\{ \frac{0.517559 - 0.516568}{0.0006033 - 0.003614} \right\} \end{array} \right]$$

$$= 0.5177575$$

So, the desired root of the equn. is 0.5177575.

Q: Find out the root approx. of $\sqrt{3}$ by using secant method upto 3 decimal places.

Ans - $x_0 = 1.7 \Rightarrow f(x_0) = -0.11$

$x_1 = 1.8 \Rightarrow f(x_1) = 0.24$

① 1st approx

$$x_2 = x_1 - \left[f(x_1) \left\{ \frac{x_1 - x_0}{f(x_1) - f(x_0)} \right\} \right]$$

$$= 1.8 - \left[0.24 \left\{ \frac{1.8 - 1.7}{0.24 + 0.11} \right\} \right]$$

$$= 1.731428$$

// -> Newton-Raphson Method -> This numerical approx. method is 1 of the most widely used numerical method for the approx evaluation of the root of the equn. $f(x) = 0$. The formula for finding the root of the equn. is given by:

Rate of convergence = 2

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad k = 1, 2, \dots$$

Let's consider the initial approx. as x_0 . Then the 1st approx will be:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\text{Similarly, } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\therefore$ This method requires 1 initial approx. to proceed. So, this is a 1 pt. form.

Q: Find the root approx of the eqn.
 $f(x) = x^4 - x - 10 = 0$ using NR method
 (Also called Newton's method) upto 4 decimal places.

Ans - Given, $f(x) = x^4 - x - 10$
 $f'(x) = 4x^3 - 1$

To find out the interval:

$$f(1) = -10 < 0$$

$$f(2) = 4 > 0$$

$\therefore$ The root lies in $(1, 2)$

$\therefore$ Let's take 1.8 as the initial approx. value $\therefore x_0 = 1.8$

① 1st approx

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.8583303475$$

② 2nd approx

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.85559085481$$

③ 3rd approx

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.8552867$$

~~1.85526675046~~

11/7 Fixed pt. iteratⁿ method. -

If a functⁿ (f) is defined on the interval $I = [a, b]$ & $f(c) = c$ for some $c \in I$, then (f) is said to have the fixed pt. (c) in the interval I . Ex: $f(x) = x^2$ has a fixed pt. 1 in the interval $(0.5, 1.5)$ & also has a fixed pt. 0 in the interval $(-0.5, 0.5)$ & has 2 fixed pts. in the interval $(-1, 2)$.

$\therefore$ A functⁿ may have 1 or more fixed pt. in any interval.

Let the equn. be $f(x) = 0$ & (a, b) be the interval in which the root of the equn. lie.

Write down the equn. $f(x)$ in form of $x = \phi(x)$, such that $|\phi'(x)| < 1$ $\forall x \in (a, b)$.

Now, starting with the no. x_0 in the interval (a, b) , we generate a sequence of nos. or iteratⁿs i.e. $x_0, x_1, x_2, \dots, x_n, \dots$ using the iteratⁿ formula

$$x_{k+1} = \phi(x_k), k \in \mathbb{W}.$$

Nos. $x_0, x_1, \dots, x_n, \dots$ are called iterates & $\phi(x)$ functⁿ is the iteratⁿ functⁿ.

This method is also called functional iteratⁿ method. & this is a 1 pt. method.

The rate of convergence is 1 i.e. linearly convergent method.

Q. Find out the root approx. of $f(x) = x^2 - 2x - 3 = 0$ using fixed pt. iteratⁿ method correct

upto 2 decimal places:

$$\text{Ans} - f(x) = x^2 - 2x - 3 = 0$$

$$\therefore \left. \begin{aligned} f(2.5) &= -1.75 < 0 \\ f(3.5) &= 2.25 > 0 \end{aligned} \right\}$$

Let's choose $(2.5, 3.5)$ as the initial approx. interval.

$$\therefore \phi_1(x) = \frac{x^2 - 3}{2}$$

$$\phi_2(x) = \sqrt{2x+3} \quad (\checkmark)$$

$$\phi_3(x) = \frac{3}{x-2}$$

$\therefore$ let's consider $\phi_2(x) = \sqrt{2x+3}$ as the iteratⁿ functⁿ.

Let's choose $x_0 = 2.5$

① 1st approx.

$$x_1 = \phi(x_0) = 2.82842712$$

② 2nd approx

$$x_2 = \phi(x_1) = 2.94225325$$

③ 3rd approx.

$$x_3 = \phi(x_2) = 2.98068893$$

④ 4th approx

$$x_4 = \phi(x_3) = 2.99356055$$

⑤ 5th approx

$$x_5 = \phi(x_4) = 2.99785274$$

$\therefore$, x_5 is the required root of the given equn. upto 2 decimal places.

//→ Numerical Solutⁿ for system of linear equns:-

Simultaneous linear equns. have great importance in the field of engg. & sci. In this field, the analysis of electronic circuits having no. of variant elements, analysis of network under sinusoidal steady state condit^{ns} & determinatⁿ of output of a chemical plant are some of the problems which depends on the solut^{ns} of system of linear equns.

We discuss the solutⁿ of (n) linear equns. with (n) variables by diff. direct & indirect (iterative) methods.

//→ Direct method→

Direct methods for solving system of linear equns. are methods that gives solut^{ns} after an amount of computatⁿ that can be specified in advance.

//→ Indirect method→

It's an indirect or iterative method. We start from an approx. to the real solutⁿ & if successful, obtain better & better approx from a computatⁿ. The cycle repeated as often as may be necessary for achieving a required accuracy.

Some indirect methods are:

- ① Gauss-Seidal method.
- ② SOR Method (Successive over relaxatⁿ)

Some direct methods are:

- ③ Doolittle's Method
- ④ Crout's Method
- ⑤ Cholesky Method

We prefer indirect methods if the convergence is rapid or if the matrices have large diagonal entries.

// → Gauss-Seidal method →

(IMP) This method is also known as Liebmann method or method of successive displacement. It's a conditⁿ of SOR method when $(w=1)$

consider the system of linear eqns:

$$a_1x + b_1y + c_1z = d_1 \quad \text{--- (1)}$$

$$a_2x + b_2y + c_2z = d_2 \quad \text{--- (2)}$$

$$a_3x + b_3y + c_3z = d_3 \quad \text{--- (3)}$$

$$\text{verify that } |a_1| > |b_1| + |c_1|$$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

then the gauss-seidal method can be used for the given system.

solving eqn. (1), (2) & (3), for finding x , y & z respectively, we've:

$$x = \frac{1}{a_1} (d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2} (d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3} (d_3 - a_3x - b_3y)$$

We can start the solutⁿ process with initial values $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ for x , y , z respectively.

(IMP) In the absence of initial approx, we can take them as $(0, 0, 0)$. We've to check $x^{(0)}$, $y^{(0)}$, $z^{(0)}$ only by guess.

then we get the 1st approx. using the formula:

$$x^{(1)} = \frac{1}{a_1} (d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2} (d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

$$z^{(1)} = \frac{1}{c_3} (d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

Thus, in this case, as soon as a new value for a variable is found, it's immediately used in the computatⁿ of the next component. If $x^{(k)}$, $y^{(k)}$, $z^{(k)}$ are the k^{th} approx, then the $(k+1)^{\text{th}}$ approx will be:

$$\begin{aligned} x^{(k+1)} &= \frac{1}{a_1} (d_1 - b_1 y^{(k)} - c_1 z^{(k)}) \\ y^{(k+1)} &= \frac{1}{b_2} (d_2 - a_2 x^{(k+1)} - c_2 z^{(k)}) \\ z^{(k+1)} &= \frac{1}{c_3} (d_3 - a_3 x^{(k+1)} - b_3 y^{(k+1)}) \end{aligned}$$

Continuing this process, we can get the required result or solutⁿ.

If $Ax = b$ is the system of linear eqns. then:

Ⓐ $A = (a_{ij})_{n \times n}$

Ⓑ $x = (x_i)_{n \times 1}$

Ⓒ $b = (b_i)_{n \times 1}$

then the gauss-seidel iteratⁿ formula will be:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} \cdot x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} \cdot x_j^{(k)} \right]$$

Q: Solve the following system of linear eqns. by using Gauss-Seidal method correct upto 3 decimal places.

$$10x - 5y - 2z = 3$$

$$x + 6y + 10z = -3$$

$$4x - 10y + 3z = -3$$

Ans - Let's rewrite the system of eqns. as:

$$10x - 5y - 2z = 3 \quad \text{--- (1)}$$

$$4x - 10y + 3z = -3 \quad \text{--- (2)}$$

$$x + 6y + 10z = -3 \quad \text{--- (3)}$$

$$\therefore |10| > |-5| + |-2|$$

$$|-10| > |4| + |3|$$

$$|10| > |4| + |6|$$

$$\therefore x = \frac{1}{10} (3 + 5y + 2z)$$

$$y = \frac{1}{-10} (-3 - 4x - 3z)$$

$$= \frac{1}{10} (3 + 4x + 3z)$$

$$z = \frac{1}{10} (-3 - x - 6y)$$

$$= -\frac{1}{10} (3 + x + 6y)$$

So, the Gauss-Seidal iteration formula for the given system of linear eqns. is given by:

$$x^{(k+1)} = \frac{1}{10} (3 + 5y^{(k)} + 2z^{(k)})$$

$$y^{(k+1)} = \frac{1}{10} (3 + 4x^{(k+1)} + 3z^{(k)})$$

$$z^{(k+1)} = -\frac{1}{10} (3 + x^{(k+1)} + 6y^{(k+1)})$$

Let's choose the initial approx. as

$$x^{(0)} = y^{(0)} = z^{(0)} = 0 \quad y^{(6)} = 0.2850393629$$

① 1st approx:

$$x^{(1)} = 0.3$$

$$y^{(1)} = 0.42$$

$$z^{(1)} = -0.582$$

$$z^{(6)} = -0.5051729963$$

So, the required solution to the given system of eqn. upto 3 decimal places is
 $x = 0.3414$; $y = 0.2850$;
 $z = -0.5051$

② 2nd approx:

$$x^{(2)} = 0.3936$$

$$y^{(2)} = 0.28284$$

$$z^{(2)} = -0.509064$$

③ 3rd approx:

$$x^{(3)} = 0.3396072$$

$$y^{(3)} = 0.28312368$$

$$z^{(3)} = -0.503834928$$

④ 4th approx:

$$x^{(4)} = 0.3407948544$$

$$y^{(4)} = 0.2851674634$$

$$z^{(4)} = -0.5051799635$$

⑤ 5th approx:

$$x^{(5)} = 0.344547739$$

$$y^{(5)} = 0.2850651066$$

$$z^{(5)} = -0.5051938379$$

⑥ 6th approx:

$$x^{(6)} = 0.3414937857$$

// $\rightarrow$ Positive Definite matrix $\rightarrow$ A symmetric matrix A with real entries is called +ve definite if the real no. $Z^T A Z > 0$ for any non-zero column matrix Z where Z^T is the transpose of Z . Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{2 \times 2} \quad \& \quad Z = \begin{bmatrix} x \\ y \end{bmatrix}_{2 \times 1} \quad x, y \in \mathbb{R}$$

$$\therefore Z^T A Z = x^2 + 4xy + 3y^2$$

This no. isn't always +ve for every non-zero column matrix Z . So, the given matrix A is not +ve definite.

If A is a symmetric matrix & A_K is the upper left $(K \times K)$ sub-matrix, & if all upper left $(K \times K)$ determinant of that matrix A are +ve, then the matrix is +ve definite. Ex:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow |3| > 0 \quad \& \quad |A| = 8 > 0$$

// $\rightarrow$ SOR Method $\rightarrow$

- ① This is a generalizatⁿ of Gauss-Seidel method.
- ② This method is often used when the coefficient matrix of the system of linear eqns. is symmetric.
- ③ Let's explain the SOR method in the case of 3 linear eqns. with 3 unknowns with relaxatⁿ parameter (ω). Then we can extend the method into (n) linear eqns. with (n) unknowns.

Mechanism:-

Consider the system of linear eqns.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The given system in matrix form can be written as $AX=b$, where:

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad b = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

We will solve the system of eqns. using SOR methods in 4 steps:

Step-1

Verify the sufficient condition for the SOR method i.e. the coefficient matrix A is symmetric +ve definite, i.e.

⊗ A is symmetric $\Rightarrow A = A^T$

⊗ A is a +ve definite matrix i.e.

$$|a_{11}| > 0; \quad \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} > 0 \quad \& \quad \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix} > 0$$

Step-2

Verify that the relaxation parameter (ω) satisfy the condition $0 < \omega < 2$.

Step-3

Write down the Gauss-Seidel iteration scheme for the given system of linear eqns:

$$x^{(k+1)} = \frac{1}{a_1} (d_1 - b_{12}y^{(k)} - c_{13}z^{(k)}) \dots \quad (1)$$

$$y^{(k+1)} = \frac{1}{b_2} (d_2 - a_{21}x^{(k+1)} - c_{23}z^{(k)}) \dots \quad (2)$$

$$z^{(k+1)} = \frac{1}{c_3} (d_3 - a_{31}x^{(k+1)} - b_{32}y^{(k+1)}) \dots \quad (3)$$

Step-4

Multiply the RHS of eqns. ①, ② & ③ by (ω) , & adding the vectors $x^{(k)}, y^{(k)}, z^{(k)}$ multiplying by $(1-\omega)$:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{a_1} (d_1 - b_{12}y^{(k)} - c_{13}z^{(k)})$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \frac{\omega}{b_2} (d_2 - a_{21}x^{(k+1)} - c_{23}z^{(k)})$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \frac{\omega}{c_3} (d_3 - a_{31}x^{(k+1)} - b_{32}y^{(k+1)})$$

continuing this process until the convergence is assured.

Q: solve the following system of linear eqns. using SOR method:-

$$3x - y + z = -1 \quad \text{--- ①}$$

$$-x + 3y - z = 7 \quad \text{--- ②}$$

$$x - y + 3z = -7 \quad \text{--- ③}$$

check that the SOR method with the value $\omega = 1.25$ as the relaxatⁿ parameter can be used to solve the given system of eqns. & then solve it performing 3 iterations.

$$\text{Ans- } A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = A^T$$

Hence, A is a symmetric matrix. to check +ve definiteness of A :

$$|3| = 3 > 0$$

$$\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8 > 0$$

$$\begin{vmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 20 > 0$$

thus, A is +ve definite. Now, $\omega \in (0, 2)$
 $\Rightarrow \omega = 1.25$. So, we can use this
 relaxatⁿ parameter in SOR method. Now,

$$x^{(k+1)} = \frac{1}{3} (-1 + y^{(k)} - z^{(k)})$$

$$y^{(k+1)} = \frac{1}{3} (7 + x^{(k+1)} + z^{(k)})$$

$$z^{(k+1)} = \frac{1}{3} (-7 - x^{(k+1)} + y^{(k+1)})$$

Now, multiplying the RHS of eqn. (1), (2) & (3)
 with (ω) & adding the vectors $x^{(k)}$, $y^{(k)}$, $z^{(k)}$
 multiplying by $(1-\omega)$ respectively, we've:

$$x^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{3} (-1 + y^{(k)} - z^{(k)})$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \frac{\omega}{3} (7 + x^{(k+1)} + z^{(k)})$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \frac{\omega}{3} (-7 - x^{(k+1)} + y^{(k+1)})$$

Let's assume the initial approx. as $x^{(0)} =$
 $y^{(0)} = z^{(0)} = 0$.

① 1st approx:

$$x^{(1)} = -0.41666666$$

$$y^{(1)} = 2.74305555$$

$$z^{(1)} = -1.60011574$$

② 2nd approx:

$$x^{(2)} = 1.4971547025$$

$$y^{(2)} = 2.188002345$$

$$z^{(2)} = -2.22878454$$

③ 3rd approx:

$$x^{(3)} = 1.04937252$$

$$y^{(3)} = 1.878244403$$

$$z^{(3)} = -2.014107242$$

// -> Direct methods for solving system of linear eqns:-

1-> Doolittle's Method:-

N.B: A sq. matrix whose elements below the principal diagonal are all zero is called an Upper triangular matrix.

N.B: An upper triangular matrix whose principal diagonal elements are 1 are called an unit upper triangular matrix.

Let's consider a system of ³ linear eqns. with 3 unknowns, i.e:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

∴ We can write the above system of linear eqns. as $\boxed{AX=b}$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now, factorise (A) into an LU decomposition where (L) is an unit lower triangular matrix & (U) is an upper triangular matrix. i.e. $\boxed{A=LU} \dots \textcircled{2}$

Solving eqn. $\textcircled{2}$, we can find:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ \& } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Putting equn. (2) in (1), we get: $LUX = b \dots (3)$

Now, let $UX = Y \dots (4)$, where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Putting equn. (4) in (3), we get: $LY = b \dots (5)$

We can find out the value of (Y) by solving equn. (5) using forward substitution method.

Now, put the value of (Y) in equn. (4). So, find the value of (X) by solving equn. (4) using backward substitution method.

Q: Solve the following system of linear eqns by using doolittle's method.

$$3x + y + z = 4$$

$$x + 2y + 2z = 3$$

$$2x + y + 3z = 4$$

Ans: The above system of linear eqns. in matrix form can be written as: $AX = b$, where:

①

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; b = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

Let, $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{21} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$\Rightarrow u_{11} = 3 ; u_{13} = 1 ; u_{12} = 1$$

$$\Rightarrow l_{21}u_{11} = 1 \Rightarrow l_{21} = 1/3$$

$$\Rightarrow l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = 5/3$$

$$\Rightarrow l_{21}u_{13} + u_{23} = 2 \Rightarrow u_{23} = 5/3$$

$$\Rightarrow l_{31}u_{11} = 2 \Rightarrow l_{31} = 2/3$$

$$\Rightarrow l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow l_{32} = 1/5$$

$$\Rightarrow l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow u_{33} = 2$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix} ; U = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix}$$

Putting equn. ② in ①, we've: $LUx = b$ --- ③

Let $UX = Y$ --- ④, where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Putting equn. ④ in ③, we get: $LY = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/5 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ \frac{y_1}{3} + y_2 \\ \frac{2y_1}{3} + \frac{y_2}{5} + y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = 4 ; \frac{y_1}{3} + y_2 = 3 \Rightarrow y_2 = 5/3$$

$$\Rightarrow \frac{2y_1}{3} + \frac{y_2}{5} + y_3 = 4 \Rightarrow y_3 = 1$$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

Now, putting the value of (Y) in eqn. (4), we've:

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5/3 & 5/3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x + y + z \\ 0 + 5y/3 + 5z/3 \\ 0 + 0 + 2z \end{bmatrix} = \begin{bmatrix} 4 \\ 5/3 \\ 1 \end{bmatrix}$$

$$\Rightarrow z = 1/2, y = 4/2, x = 1$$

// $\rightarrow$ Cramer's Method:-

Let's consider the system of ³ linear eqns:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form, the above system can be written as: $AX = b \dots \textcircled{1}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now, factorise the coefficient matrix (A) into "LU decomposition", where:

L = Lower triangular matrix.

U = Unit upper triangular matrix.

$$\therefore A = LU^* \dots \textcircled{2}$$

Putting $\textcircled{2}$ in $\textcircled{1}$, we've $LUX = b \dots \textcircled{3}$

Let $UX = Y \dots (4)$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Putting (4) in (3), we've: $LY = b \dots (5)$

Solving eqn. (5) using forward substitutⁿ method, we get the value of (Y) .

Then putting the value of (Y) in eqn. (4), we can find the value of (X) using backward substitutⁿ method.

// Cholesky Method: - Let's consider the system of 3 linear eqns:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form, the above system:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\therefore AX = b \dots (1)$$

Now, factorize A into LL^T decompositⁿ where, L is a lower triangular matrix & L^T is its transpose.

$$\therefore A = LL^T \dots (2)$$

Putting (2) in (1), we've: $LL^T X = b \dots (3)$

Let $L^T X = Y$, where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \dots (4)$

Putting (4) in (3), we've $LY = b \dots (5)$

Solving (5) using forward substitutⁿ, we can get the value of (Y) .

Putting the value of (Y) in eqn. we

can get the value of (x) using backward substitutⁿ.

// $\rightarrow$ Interpolatⁿ $\rightarrow$ the method of estimating unknown values with the help of given set of tabular pts. or nodes & corresponding set of tabular values is known as interpolatⁿ.

If the unknown lies in the range of the given tabular pts, then the method is called as interpolatⁿ. & if it lies outside the tabular pts. it's called extrapolatⁿ.

Let the explicit nature of a functⁿ $f(x)$ isn't known, but the value of the functⁿ at $(n+1)$ distinct pts. $x_0, x_1, x_2, \dots, x_n$ where $x_0 < x_1 < \dots < x_n$ is known. Then the process of interpolatⁿ is to find out another functⁿ $P(x)$ of degree (n) , such that: $P(x_i) \simeq f(x_i) \forall i = 0 \text{ to } n$. This $P(x)$ is called the interpolating polynomial of functⁿ $f(x)$.

Here $|P(x_i) - f(x_i)| < \epsilon$, where (ϵ) is a small +ve integer. That's why, we can call that error as zero tolerance error.

N.B: The interpolating polynomial of a functⁿ is unique.

There are several types of interpolating polynomial. They are named as:

- ① Newton's forward diff. IP (NFDIP)
- ② Newton's backward diff. IP (NBDIP)
- ③ Newton's divided diff. IP (NDDIP)
- ④ Lagrange's IP.

// $\rightarrow$ Lagrange's IP :- Let the nodes are $x_0, x_1, \dots, x_n$ in an interval $[a, b]$ &

the corresponding functional values are $f(x_0)$, $f(x_1)$, ..., $f(x_n)$, then the Lagrange's IP is given by:

$$P_n(x) = \sum_{i=0}^n l_i(x) f(x_i), \text{ where:}$$

$$l_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

† $l_i(x)$ is called Lagrangian function.

N.B: $l_i(x_i) = 1$

$$l_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Q: Find out the interpolating polynomial using the given data by Lagrange's IP.

x	-1	0	1
$f(x)$	2	1	3

Ans - Let $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$\Rightarrow f(x_0) = 2$, $f(x_1) = 1$, $f(x_2) = 3$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2} = \frac{x^2-x}{2}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = 1-x^2$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{(1+1)(1-0)} = \frac{x(x+1)}{2}$$

So, the Lagrange's IP is given by:

$$P_2(x) = \sum_{i=0}^2 l_i(x) f(x_i)$$

$$\begin{aligned}
 &= l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) \\
 &= \frac{x(x-1)}{2} \cdot (-2) + (1-x^2) \cdot 1 + \frac{x(x+1)}{2} \cdot (3) \\
 &= \frac{1}{2} (3x^2 + x + 2)
 \end{aligned}$$

11 → Newton's divided diff. IP:-

Let the set of tabular pts. & its corresponding functional values are given by $x_0, x_1, \dots, x_n$ & $f_0, f_1, \dots, f_n$. Then, the divided diff. of:

① 0th order is defined by:

$$f[x_0] = f(x_0) = f_0$$

② 1st order divided diff:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

③ 2nd order divided diff:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

④ 3rd order divided diff:

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

∴ In general, nth order divided diff. is given by:

$$f[x_0, x_1, x_2, \dots, x_n] = \frac{f[x_1, x_2, x_3, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

11 → Derivatⁿ of NDDIP:-

$$f[x, x_0] = \frac{f(x_0) - f(x)}{x_0 - x}$$

$$\Rightarrow f(x) = (x - x_0) \cdot f[x, x_0] + f(x_0) \quad \text{①}$$

$$\text{Also: } f[x, x_0, x_1] = \frac{f[x_0, x_1] - f[x, x_0]}{x_1 - x_0}$$

$$\Rightarrow f[x, x_0] = f[x_0, x_1] + (x - x_1)f[x, x_0, x_1] \quad \text{--- (2)}$$

$\therefore$ Putting (2) in (1), we've:

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) \{ f[x_0, x_1] + (x - x_1) \\ &\quad \cdot f[x, x_0, x_1] \} \\ &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0) \cdot \\ &\quad \cdot (x - x_1) f[x, x_0, x_1] \end{aligned}$$

Now, proceeding in this way, after n th divided difference, we get:

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0) \cdot \\ &\quad \cdot (x - x_1) f[x_0, x_1, x_2] + \dots + \\ &\quad (x - x_0)(x - x_1) \dots (x - x_n) f[x, x_0, x_1, \dots, \\ &\quad \dots, x_n] \end{aligned}$$

Let $P_n(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0) \cdot (x - x_1) f[x_0, x_1, x_2] + \dots + (x - x_0) \cdot (x - x_1) \dots (x - x_{n-1}) f[x_0, x_1, \dots, x_n]$ AN

Error in NDDIP is:

$$E(x) = |f(x) - P_n(x)| = (x - x_0)(x - x_1) \dots (x - x_n) f[x, x_0, x_1, \dots, x_n]$$

$\therefore P_n(x)$ is a polynomial of degree (n) is called NDDIP.

$$\therefore f(x) = P_n(x) + E(x)$$

NB: (1) $P_0(x) = f(x_0)$

$$\begin{aligned} \text{(2) } P_1(x) &= f(x_0) + (x - x_0) f[x_0, x_1] \\ &= P_0(x) + (x - x_0) f[x_0, x_1] \end{aligned}$$

$$\text{(3) } P_2(x) = P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

In general, we get:

$$P_n(x) = P_{n-1}(x) + (x - x_0)(x - x_1) \dots (x - x_{n-1}) \cdot f[x_0, x_1, \dots, x_n]$$

11 -> Divided difference table: -

x	$f(x)$	1st order	2nd order	3rd order	4th order
x_0	f_0	$f[x_0, x_1]$			
x_1	f_1		$f[x_0, x_1, x_2]$		
x_2	f_2	$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
x_4	f_4	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	

Q:

x	-1	0	1
$f(x)$	2	1	3

Ans -

x	$f(x)$	1st Ord.	2nd Order
-1	2		
		-1	
0	1		1.5
		2	
1	3		

$$\begin{aligned}
 \therefore P_2(x) &= f(x_0) + (x - x_0)f[x_0, x_1] \\
 &\quad + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \\
 &= 2 + (x + 1)(-1) + (x + 1)(x - 0) \cdot 1.5 \\
 &= 2 - (x + 1) + 1.5(x^2 - 0 + x - 0) \\
 &= 1.5x^2 + 0.5x + 1
 \end{aligned}$$

Q:

x	$f(x)$
8	2.079442
9	2.197225
9.5	2.251292
11	2.397895

Find $f(9.2)$

<u>Ans</u> x	$f(x)$	1st	2nd	3rd
8	2.079442	0.147783		
9	2.497225	0.408137	-0.006432	
9.5	2.251292		-0.005199	0.000410
11	2.397895	0.097735		

$$\therefore P_3(9.2) = f(x_0) + (9.2 - x_0) f[x_0, x_1] \\ + (9.2 - x_0)(9.2 - x_1) f[x_0, x_1, x_2] \\ + (9.2 - x_0)(9.2 - x_1)(9.2 - x_2) \\ \cdot f[x_0, x_1, x_2, x_3]$$

$$\Rightarrow P_3(9.2) = 2.2192084$$

$$\therefore f(9.2) \simeq 2.2192084 \simeq \ln(9.2)$$

// Newton's forward diff. IP:-

(IMP) Forward differences are usually more convenient than the divided differences in case the nodes are equally spaced.

Hence, for having the forward differences, we consider or assume the nodes to be equally spaced.

// Forward difference $\rightarrow$ Suppose the function $y = f(x)$ has the values $y_0, y_1, y_2, \dots, y_n$ for the values of $x = x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$. $\Rightarrow x_n = x_0 + nh$, where the nodes are equally spaced by difference (h) . Here, (h) is called step length. Then the forward diff. operator which is denoted as (Δ) del, is defined as $\Delta f(x) = f(x+h) - f(x)$. The quantity $\Delta f(x)$ is called the 1st forward diff. of $f(x)$ & we write $\Delta^0 f(x) = f(x)$ by

convenience we convert ^{n} of for $n > 1$, the n -th order forward diff. is defined by $\Delta^n f(x) = \Delta(\Delta^{n-1} f(x))$. $\therefore$ For $n=2$:

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$\Rightarrow \Delta^2 f(x) = \Delta(f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

$$\therefore \Delta^2 f(x_0) = f(x_0+2h) - 2f(x_0+h) + f(x_0)$$

$$= f(x_2) - 2f(x_1) + f(x_0)$$

$$\Rightarrow \Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$\therefore$ In general, we can also write:

$$\Delta y_k = y_{k+1} - y_k \Rightarrow \Delta f(x_k) = f(x_{k+1}) - f(x_k)$$

$$\therefore \Delta^2 y_k = \Delta(\Delta y_k) = \Delta(y_{k+1} - y_k) = \Delta y_{k+1} - \Delta y_k$$

$$\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$$

Q: Show that $\sum_{k=0}^{n-1} \Delta y_k = y_n - y_0$

$$\begin{aligned} \text{Ans: } \sum_{k=0}^{n-1} \Delta y_k &= \Delta y_0 + \Delta y_1 + \Delta y_2 + \Delta y_3 + \dots + \Delta y_{n-1} \\ &= y_1 - y_0 + y_2 - y_1 + y_3 - y_2 + y_4 - y_3 + \dots \\ &\quad + y_n - y_{n-1} \\ &= y_n - y_0 \end{aligned}$$

$\Rightarrow$ Forward diff. table: -

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	y_0	$\Delta f(x_0)$ $= \Delta y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	
x_2	y_2	Δy_2		
x_3	y_3			

Q: Let the functⁿ $f(x) = x^2 + 2x + 1$ & set of pts. be 0, 1, 2, 3, 4. Write down the forward diff. table for the given data.

Ans-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1	3	2	0	0
1	4	5	2	0	
2	9	7	2		
3	16	9			
4	25				

// Theorem :-

Let $x_i = x_0 + ih, i = 0(1)n$, are equally spaced nodes with step length (h) . Then:

$$f[x_0, x_1, \dots, x_n] = \frac{\Delta^n f(x_0)}{n! h^n}$$

Proof: Let's prove the th^m by using method of inductⁿ.

$\therefore$ for, $n=1$:

$$\begin{aligned} f[x_0, x_1] &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h} \\ &= \frac{\Delta f(x_0)}{1! h^1} \end{aligned}$$

$\therefore$ It's true for $K=1$.

Let's assume that the given thing is true for $K=n-1$, i.e. $f[x_0, x_1, \dots, x_{n-1}] = \frac{\Delta^{n-1} f(x_0)}{(n-1)! h^{n-1}}$

Now, we've to prove that the above thing for $K=n$:

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}$$

$$= \frac{\Delta^{n-1} f(x_1)}{(n-1)! h^{n-1}} - \frac{\Delta^{n-1} f(x_0)}{(n-1)! h^{n-1}}$$

$$= \frac{1}{(n-1)! nh^n} \left[\Delta^{n-1} f(x_1) - \Delta^{n-1} f(x_0) \right]$$

$$= \frac{1}{n! h^n} \left[\Delta^n f(x_0) \right] = \frac{\Delta^n f(x_0)}{n! h^n}$$

We know the NDDIP is:

$$f(x) \approx P(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0) \cdot (x-x_1) f[x_0, x_1, x_2] + \dots + (x-x_0) \cdot (x-x_1) \cdot \dots \cdot (x-x_{n-1}) \cdot f[x_0, \dots, x_n]$$

From the above th^m, we know that:

$$f[x_0, \dots, x_n] = \frac{\Delta^n f(x_0)}{n! h^n}$$

So, putting these values, we get:

$$f(x) \approx P(x) = f(x_0) + (x-x_0) \frac{\Delta f(x_0)}{1! h^1} + (x-x_0) \cdot (x-x_1) \frac{\Delta^2 f(x_0)}{2! h^2} + \dots + (x-x_0) \cdot (x-x_1) \cdot \dots \cdot (x-x_{n-1}) \cdot \frac{\Delta^n f(x_0)}{n! h^n}$$

→ This formula is called NFDIP

NB: Let $x_i = x_0 + ih$, $i = 0(1)n$

If $\Delta^n f(x)$ is a constant for a tabulated function $f(x)$, then $f(x)$ is a polynomial of degree (n) .

Q: Find the degree of the polynomial from the given data using forward diff. table:

x	-2	-1	0	1	2	3
$f(x)$	-5	1	1	1	7	25

Ans-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
-2	-5	6	8	12	26	90
-1	1	2	4	14	64	
0	1	2	10	50		
1	1	8	40			
2	7	32				
3	25					

Q! For the following data, calculate the differences & obtain NFDIP of the data as are!

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

Ans- Here, $h=1$

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	3	3	2	0
1	6	5	2	0
2	11	7	2	
3	18	9		
4	27			

$$\begin{aligned}
 \therefore P(x) &= f(x_0) + (x-x_0) \frac{\Delta f(x_0)}{1! h^1} + (x-x_0)(x-x_1) \frac{\Delta^2 f(x_0)}{2! h^2} \\
 &= 3 + (x-0) \times \frac{3}{1 \times 1} + (x-0)(x-1) \times \frac{2}{2 \times 1} \\
 &= x^2 + 2x + 3
 \end{aligned}$$

Q! Find $f(0.02)$ using the given data by NFDIP:-

x	0	0.1	0.2	0.3	0.4
$f(x)$	1	1.1052	1.2214	1.3499	1.4918

Ans-

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	1	0.1052	0.011	0.0013	-0.0002
0.1	1.1052	0.1162	0.0123	0.0011	

0.2	1.2214	0.1285	0.0134
0.3	1.3499	0.1419	
0.4	1.4918		

$$\therefore P(x) = 1 + (x-0) \frac{0.1052}{1!h} + (x-0)(x-0.1) \frac{0.011}{2!h^2} + (x-0)(x-0.1)(x-0.2) \frac{0.0013}{3!h^3} + (x-0)(x-0.1)(x-0.2)(x-0.3) \frac{(0-0.0002)}{4!h^4}$$

$$\therefore f(0.02) = 1.02022912$$

Q:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	0	1.17365	-2.00528	2.99489	-3.99969
10	1.17365	-0.8316 ₃	0.98961	-1.0048	
20	0.34202	0.15798	-0.01519		
30	0.5	0.14279			
40	0.64279				

$$\therefore f(38) \approx 0.798067184$$

N.B. When the forward differences in the forward diff. table are rapidly increasing, in this case, the interpolating polynomial doesn't fit to the functⁿ at all. (IMP)

// NBDIP $\rightarrow$ For a real valued functⁿ (f) of real variable (x), the backward diff. operator denoted by ∇ (abla) is defined to be $\nabla f(x) = f(x) - f(x-h)$, where (h) is the step length, which is const. The quantity $\nabla f(x)$ is called the 1st backward diff. of $f(x)$. We define the 2nd order backward diff. of $f(x)$ by $\nabla^2 f(x) = \nabla(\nabla f(x))$. In general, the n th order backward diff. is defined by $\nabla^n f(x) = \nabla \{ \nabla^{n-1} f(x) \}$.

N.B. ① $\nabla f(x_k) = f(x_k) - f(x_{k-1})$

$$\textcircled{A} \nabla^2 f(x_k) = \nabla f(x_k) - \nabla f(x_{k-1})$$

$$\textcircled{A} \nabla^n f(x_k) = \nabla^{n-1} f(x_k) - \nabla^{n-1} f(x_{k-1})$$

$$\therefore$$

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	$f(x_0)$				
x_1	$f(x_1)$	$\nabla f(x_1)$			
x_2	$f(x_2)$	$\nabla f(x_2)$	$\nabla^2 f(x_2)$		
x_3	$f(x_3)$	$\nabla f(x_3)$	$\nabla^2 f(x_3)$	$\nabla^3 f(x_3)$	
x_4	$f(x_4)$	$\nabla f(x_4)$	$\nabla^2 f(x_4)$	$\nabla^3 f(x_4)$	$\nabla^4 f(x_4)$

Q: Construct a backward diff. table for the given data: -

x	1	2	3	4
$f(x)$	63	52	43	38

Ans -

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
1	63			
2	52	-11		
3	43	-9	2	
4	38	-5	4	2

//> Relatⁿ between forward & backward diff. operator: -

$$\textcircled{A} \nabla f(x_n) = f(x_n) - f(x_{n-1}) = \Delta f(x_{n-1})$$

$$\therefore \nabla f_n = \Delta f_{n-1}$$

$$\textcircled{B} \nabla^2 f(x_n) = \nabla(\nabla f(x_n))$$

$$= \nabla(f(x_n) - f(x_{n-1}))$$

$$= \nabla f(x_n) - \nabla f(x_{n-1})$$

$$= \Delta f(x_{n-1}) - \Delta f(x_{n-2})$$

$$= \Delta^2 f(x_{n-2})$$

$$\therefore \nabla^2 f(x_n) = \Delta^2 f(x_{n-2})$$

$$\Rightarrow \nabla^K f(x_n) = \Delta^K f(x_{n-K})$$

$$\Rightarrow \boxed{\nabla^K f_n = \Delta^K f_{n-K}}$$

// $\rightarrow$ True NDDIP $\rightarrow$ we know the NDDIP for the nodes $x_0, \dots, x_n$ such that the nodes will be equally spaced: $x_i = x_0 + ih$, $i = 0(1)n$ is given by:

$$f(x) = f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + \dots + (x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)f[x_0, x_1, x_2, \dots, x_n]$$

The same formula for nodes in order $x_n, x_{n-1}, x_{n-2}, \dots, x_1, x_0$ is given by:

$$f(x) = f(x_n) + (x-x_n)f[x_n, x_{n-1}] + (x-x_n)(x-x_{n-1})f[x_n, x_{n-1}, x_{n-2}] + \dots + (x-x_n)(x-x_{n-1})\dots(x-x_1)f[x_n, \dots, x_0]$$

According to the th^m, $f[x_n, \dots, x_0]$ divided diff. is: ①

$$\boxed{f[x_n, \dots, x_0] = \frac{\nabla^n f(x_n)}{n! h^n}}$$

Putting those values in equn. ①, we get:

$$f(x) = f(x_n) + (x-x_n) \frac{\nabla f(x_n)}{1! h^1} + (x-x_n)(x-x_{n-1}) \frac{\nabla^2 f(x_n)}{2! h^2} + \dots + (x-x_n)\dots(x-x_1) \frac{\nabla^n f(x_n)}{n! h^n}$$

Q: For the following data, find the NBIP & value of $f(0.35)$.

Ans -

x	$f(x)$	∇f	$\nabla^2 f$	~~$\nabla^3 f$~~	$\nabla^4 f$
0	1				
0.1	1.1052	0.1052			
0.2	1.2214	0.1162	0.011		
0.3	1.3499	0.1285	0.0123	0.0013	
0.4	1.4918	0.1419	0.0134	0.0014	-0.0002

$$\therefore P(0.35) \simeq f(0.35) \simeq 1.4191$$